A Kähler-Einstein inspired anzatz for $Spin(7)$ holonomy metrics and its solution

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Abstract

We construct propose an anzatz for $Spin(7)$ metrics as an R bundle over closed G_2 structures. These G_2 structures are R^3 bundles over 4-dimensional compact quaternion Kähler spaces. The inspiration for the anzatz metric comes from the Bryant-Salamon construction of G_2 holonomy metrics and from the fact that the twistor space of any compact quaternion Kähler space is Kähler-Einstein. The reduction of the holonomy to a subgroup of $Spin(7)$ gives non linear system relating three unknown functions of one variable. We obtain a particular solution and we find that the resulting metric is a Calabi-Yau cone over an Einstein-Sassaki manifold which means that the holonomy is reduced to $SU(4) \subset Spin(7)$. Another coordinate change show us that our metrics are hyperkahler cones known as Swann bundles, thus the holonomy is reduced to $Sp(2) \subset SU(4) \subset Spin(7)$ and the cone is tri-Sassakian. We revert our argument and state that the Swann bundle define a closed G_2 structure by reduction along an isometry. We calculate the torsion classes for such structure explicitly.

1. Introduction

There exist a growing interest in the construction of $Spin(7)$ holonomy metrics due to their application in supergravity compactification preserving certain amount of supersymmetry [13]-[30]. The present work is concerned with this task and from the analysis performed here it is obtained the following proposition.

Proposition *Let us consider a compact quaternion Kähler space M in $d = 4$ with metric g_q and with cosmological constant Λ normalized to 3. For any of such metrics there always exist a basis e^a such that $g_q = \delta_{ab} e^a \otimes e^b$ for which the $Sp(1)$ part of the spin connection ω_-^a and the negative oriented Kähler triplet $\overline{J}_i$ defined by*

$$\omega_-^a = \omega_0^a - \epsilon_{abc}\omega_c^b, \qquad \overline{J}_1 = e^1 \wedge e^2 - e^3 \wedge e^4,$$

$$\overline{J}_2 = e^1 \wedge e^3 - e^4 \wedge e^2 \qquad \overline{J}_3 = e^1 \wedge e^4 - e^2 \wedge e^3,$$

satisfy the relations

$$d\omega_-^i + \epsilon_{ijk}\omega_-^j \wedge \omega_-^k = \overline{J}_i, \qquad d\overline{J}^i = \epsilon_{ijk}\overline{J}^j \wedge \omega_-^k. \tag{1.1}$$

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Let τ and u_i be four new coordinates, $u=\sqrt{u^i u^i}$, $\alpha_i = du^i + \epsilon^{ijk}\omega_-^j u^k$ and H a τ-independent one form. Then the 8-dimensional metric

$$g_8 = \frac{(dt+H)^2}{e^{\frac{3}{2}h}} + e^{2f+\frac{1}{2}h}\alpha_i\alpha_i + e^{2g+\frac{1}{2}h} g_q \tag{1.2}$$

together with the four form

$$\Phi_4 = (dt+H)\wedge\Big(e^{3f}\alpha_1\wedge\alpha_2\wedge\alpha_3 + e^{f+2g}\alpha_i\wedge\overline{J}_i\Big) + e^{2(f+g)+h}\frac{\epsilon_{ijk}}{2}\alpha_i\wedge\alpha_j\wedge\overline{J}_k + e^{4g+h}e_1\wedge e_2\wedge e_3\wedge e_4,$$

constitute an $Spin(7)$ structure preserved by the Killing vector ∂_τ. Moreover if f, g and h are functions of u related by

$$ue^{3f} = (e^{f+2g})', \qquad \lambda(e^{3f} - \frac{e^{f+2g}}{u^2}) = (e^{2(f+g)+h})',$$

$$4ue^{2(f+g)+h} - 2\lambda e^{f+2g} = (e^{4g+h})',$$

and the 1-form H satisfy

$$dH = -\widetilde{u}_i\overline{J}_i + \frac{\epsilon_{ijk}}{2}\widetilde{u}_i\theta_j\wedge\theta_k, \tag{1.3}$$

being $\theta_i = d(\widetilde{u}^i) + \epsilon^{ijk}\omega_-^j\widetilde{u}^k$ and $\widetilde{u}^i = u^i/u$, then Φ_4 will be closed and therefore, the holonomy of the metric (2.29) will be included in $Spin(7)$.

We will show below that the integrability condition for (1.3), namely

$$d(\widetilde{u}_i\overline{J}_i - \frac{\epsilon_{ijk}}{2}\widetilde{u}_i\theta_j\wedge\theta_k) = 0$$

is always satisfied due to the fact that the twistor space Z of any compact quaternion Kähler space M carries a Kähler-Einstein metric of positive scalar curvature [1] and the left hand side of (1.3) is the Kähler form for such metric. Also, the closure of Φ_4 follows directly from the formulas (2.13) given below. By construction, the vector field ∂_τ is Killing and if the quaternion Kähler basis possess an isometry group G which preserves the forms ω_-^i, then G will be an isometry of g_8. The construction of $Spin(7)$ holonomy metrics that follows from the proposition can also be applied to quaternion Kähler orbifolds.

2. Closed G_2 and $Spin(7)$ structures and Kähler-Einstein metrics

Quaternion Kähler spaces in brief

A key ingredient in order to construct the metrics (2.29) are quaternion Kähler manifolds and so, it is convenient to give a brief description of their properties. By definition, a quaternion Kähler space M is an euclidean $4n$ dimensional space with holonomy group Γ included into the Lie group $Sp(n)\times Sp(1)\subset SO(4n)$ [4]-[6]. This affirmation is non trivial if $D>4$, but in $D=4$ we have the well known isomorphism $Sp(1)\times Sp(1)\simeq SU(2)_L\times SU(2)_R\simeq SO(4)$ and so to state that $\Gamma\subseteq Sp(1)\times Sp(1)$ is the same that to state that $\Gamma\subseteq SO(4)$. The last affirmation is trivially satisfied for any oriented space and gives almost no restrictions about the space, therefore the definition of quaternion Kähler spaces should be modified in $d = 4$. Their main properties are the following.

- There exists three automorphism J^i ($i = 1$,2, 3) of the tangent space TM_x at a given point x with multiplication rule $J^i \cdot J^j = -\delta_{ij} + \epsilon_{ijk} J^k$, and for which the metric g_q is quaternion hermitian, that is

$$g_q(X, Y) = g(J^i X, J^i Y), \tag{2.4}$$

being X and Y arbitrary vector fields.

- The automorphisms J^i satisfy the fundamental relation

$$\nabla_X J^i = \epsilon_{ijk} J^j \omega_-^k, \tag{2.5}$$

with ∇_X the Levi-Civita connection of M and ω_-^i its $Sp(1)$ part. As a consequence of hermiticity of g, the tensor $\overline{J}^i_{ab} = (J^i)_a^c g_{cb}$ is antisymmetric, and the associated 2-form

$$\overline{J}^i = \overline{J}^i_{ab} e^a \wedge e^b$$

satisfies

$$d\overline{J}^i = \epsilon_{ijk} \overline{J}^j \wedge \omega_-^k, \tag{2.6}$$

being d the usual exterior derivative.

- Corresponding to the $Sp(1)$ connection we can define the 2-form

$$F^i = d\omega_-^i + \epsilon_{ijk} \omega_-^j \wedge \omega_-^k.$$

Then for a quaternion Kähler manifold

$$R_-^i = 2n\kappa \overline{J}^i, \tag{2.7}$$

$$F^i = \kappa \overline{J}^i, \tag{2.8}$$

being Λ certain constant and κ the scalar curvature. The tensor R_-^a is the $Sp(1)$ part of the curvature. The last two conditions implies that g is Einstein with non zero cosmological constant, i.e, $R_{ij} = 3\kappa (g_q)_{ij}$ being R_{ij} the Ricci tensor constructed from g_q. Notice that (2.8) is equivalent to (1.1) if we choose the normalization $\kappa = 1$.

- For any quaternion Kähler space the $(0, 4)$ and $(2, 2)$ tensors

$$\Theta = \overline{J}^1 \wedge \overline{J}^1 + \overline{J}^2 \wedge \overline{J}^2 + \overline{J}^3 \wedge \overline{J}^3,$$

$$\Xi = J^1 \otimes J^1 + J^2 \otimes J^2 + J^3 \otimes J^3$$

are globally defined and covariantly constant with respect to the usual Levi Civita connection.

- Any quaternion Kähler space is orientable.

- In four dimensions the Kähler triplet $\overline{J}_2$ and the one forms ω_-^a are

$$\omega_-^a = \omega_0^a - \epsilon_{abc} \omega_c^b, \qquad \overline{J}_1 = e^1 \wedge e^2 - e^3 \wedge e^4,$$

$$\overline{J}_2 = e^1 \wedge e^3 - e^4 \wedge e^2 \qquad \overline{J}_3 = e^1 \wedge e^4 - e^2 \wedge e^3.$$

In this dimension quaternion Kähler spaces are defined by the conditions (2.8) and (2.7). This definition is equivalent to state that quaternion Kähler spaces are Einstein and with self-dual Weyl tensor.

The twistor space of a quaternion Kähler space

Another very important property about compact quaternion Kähler spaces is that its twistor space is *Kähler-Einstein.* In order to define the twistor space let us note that any linear combination of the form $J = \widetilde{u}_i J_i$ is an almost complex structure on M, and the metric g_q is hermitic with respect to it. Here we have defined the scalar fields $\widetilde{u}^i = u^i/u$ and it is evident that they are constrained by the condition $\widetilde{u}^i\widetilde{u}^i = 1$. This means that the bundle of almost complex structures over M is parameterized by points on the two sphere S^2. This bundle is known as the twistor space Z of M. The space Z is endowed with the metric

$$g_6 = \theta_i\theta_i + g_q, \tag{2.9}$$

where we have defined

$$\theta_i = d(\widetilde{u}^i) + \epsilon^{ijk}\omega_-^j\widetilde{u}^k.$$

The metric (2.9) is six dimensional due to the constraint $\widetilde{u}^i\widetilde{u}^i = 1$. Corresponding to this metric we have the Kähler two form

$$\overline{J} = \widetilde{u}_i\overline{J}_i - \frac{\epsilon_{ijk}}{2}\widetilde{u}_i\theta_j \wedge \theta_k. \tag{2.10}$$

It has been proved in [1] that J is integrable and $\overline{J}$ is closed (see also [3]), therefore J is truly a complex structure and g_6 is *Kähler.* The calculation of the Ricci tensor of g_6 shows that it is also Einstein, therefore the space Z is *Kähler-Einstein.* We are using the normalization $\kappa = 1$ here, for other normalization certain coefficients must be included in (2.10). Let us introduce the covariant derivative

$$\alpha_i = du^i + \epsilon^{ijk}\omega_-^j u^k, \tag{2.11}$$

which is related to θ_i by

$$\theta^i = \frac{\alpha_i}{u} - \frac{u_i du}{u^2}.$$

With the help of this relation and the definition (2.10) it follows that

$$\begin{aligned}\overline{J} &= \frac{u_i}{u}\overline{J}_i - \frac{\epsilon_{ijk}}{2}u_i\frac{\alpha_j \wedge \alpha_k}{u^3} - \epsilon_{ijk}u_i u_j\frac{\alpha_k \wedge du}{u^4} \\ &= \frac{u_i}{u}\overline{J}_i - \frac{\epsilon_{ijk}}{2}u_i\frac{\alpha_j \wedge \alpha_k}{u^3}.\end{aligned} \tag{2.12}$$

The last expression will be needed in the following, although it is completely equivalent to (2.10). Also, the following formulae

$$\overline{J}_i \wedge \overline{J}_j = -2\delta_{ij} e_1 \wedge e_2 \wedge e_3 \wedge e_4,$$

$$d\alpha_i = \epsilon_{ijk}(u_j\overline{J}_k + \alpha_k \wedge \omega_j^-),$$

$$d(u^i u^i) = d(u^2) = 2udu = 2u^i\alpha_i,$$

$$d(\epsilon_{ijk}\alpha_i \wedge \alpha_j \wedge \alpha_k) = -udu \wedge \alpha_i \wedge \overline{J}_i, \tag{2.13}$$

$$d(\alpha_i \wedge \overline{J}_i) = 0,$$
$$d(e^{3f}) \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = (e^{3f})' du \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = 0,$$

relating $\overline{J}_i$ and α_i will be useful for our purposes. [1] For instance, the closure of (2.12) is a direct consequence of the second (2.13) together with (2.6).

A proof of the proposition

Let us go back to our task of constructing the metric (2.29). Our starting point is an eight-dimensional metric anzatz of the form

$$g_8 = \frac{(dt + H)^2}{e^{\frac{3}{2}h}} + e^{\frac{1}{2}h} g_7, \tag{2.14}$$

being g_7 a metric over a 7-manifold Y and h a function over Y. Neither the one form H nor the function h depends on t, therefore the vector field ∂_t is, by construction, Killing. Associated to the metric (2.14) we can construct the octonionic 4-form

$$\Phi_4 = (dt + H) \wedge \Phi + e^h * \Phi, \tag{2.15}$$

being Φ a G_2 invariant three form corresponding to the metric g_7 and $*\Phi$ its dual. The precise form for Φ will be found below. If we impose the condition $d\Phi_4 = 0$ then the metric g_8 will have $Spin(7)$ holonomy. [2] We will suppose that the seven dimensional metric g_7 is of the form

$$g_7 = e^{2f} \alpha_i \alpha_i + e^{2g} g_q, \tag{2.16}$$

being g_q a quaternion Kähler metric in $d = 4$ and α_i defined in (2.11). The functions f, g will depend only on the "radius" $u = \sqrt{u^i u^i}$. The form (2.16) for the 7-metric is well known and is inspired in the Bryant-Salamon construction for G_2 holonomy metrics [2]. For a metric with G_2 holonomy we have $d\Phi = d * \Phi = 0$ but we will not suppose that the holonomy of (2.16) is G_2, as in the Bryant-Salamon case. Instead we will consider 7-spaces for which the form Φ is closed but not co-closed. These are known as *closed G_2 structures* [8]-[12]. In this case the $Spin(7)$ holonomy condition $d\Phi_4 = 0$ for (2.15) will reduce to

$$dH \wedge \Phi = -d(e^h * \Phi). \tag{2.17}$$

We will find below a suitable H and g_7 for which (2.17) gets simplified even more. It seems reasonable for us to choose H in (2.15) such that

$$dH = -\lambda \overline{J}. \tag{2.18}$$

The reason for this election is that the integrability condition $d\overline{J} = 0$ will be automatically satisfied because, as we have seen above, the two form $\overline{J}$ is the Kähler form of a Kähler-Einstein metric. Here λ is a parameter, and the minus sign was introduced by convenience. Also, by selecting the basis

$$\widetilde{e}_i = e^f \alpha_i, \qquad i = 1, 2, 3 \qquad \widetilde{e}_\alpha = e^g e_\alpha \qquad \alpha = 1, 2, 3, 4,$$

[1] A more complete account of formulae can be found, for instance, in [31].

[2] The converse of this affirmation is not true, that is, the holonomy group of g_8 could be $Spin(7)$ and (2.15) could be not closed. The form (2.15) is preserved by the Killing vector, but there could exist cases for which this simplifying condition do not hold and the holonomy is still $Spin(7)$. In such cases there will exist another closed 4-form Φ_4 which is not preserved by ∂_t.

for (2.16), we can construct the G_2 invariant three form

$$\Phi = c_{abc}\widetilde{e}^a \wedge \widetilde{e}^b \wedge \widetilde{e}^c = e^{3f}\alpha_1 \wedge \alpha_2 \wedge \alpha_3 + e^{f+2g}\alpha_i \wedge \overline{J}_i \tag{2.19}$$

and its dual

$$*\Phi = e^{2(f+g)}\frac{\epsilon_{ijk}}{2}\alpha_i \wedge \alpha_j \wedge \overline{J}_k + e^{4g}e_1 \wedge e_2 \wedge e_3 \wedge e_4. \tag{2.20}$$

Here e^α is a basis for the quaternion Kähler metric g_q. As we stated above, we will consider 7-spaces for which the form Φ is closed but not co-closed. In other words we will have that $d\Phi = 0$ but $d * \Phi \neq 0$. We also suppose that f, g and h are functions of the radius u only. By using (2.13) it follows that the closure condition $d\Phi = 0$ for (2.19) leads to the equation

$$ue^{3f} = (e^{f+2g})', \tag{2.21}$$

where the tilde implies the derivation with respect to u. This is one of the equations that we need.

By another side, with the election (2.18) for H and using that $d\Phi = 0$ it follows from (2.15) that

$$d\Phi_4 = dH \wedge \Phi + d(e^h * \Phi) = -\lambda\overline{J} \wedge \Phi + d(e^h * \Phi) \tag{2.22}$$

and therefore the $Spin(7)$ holonomy condition $d\Phi_4 = 0$ is equivalent to

$$\lambda\overline{J} \wedge \Phi = d(e^h * \Phi). \tag{2.23}$$

From (2.19) and (2.12) we see that the left side of (2.23) is

$$\lambda\overline{J} \wedge \Phi = \lambda(\frac{e^{3f}}{u} - \frac{e^{f+2g}}{u^3})u^i\overline{J}^i \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - 2\lambda\frac{e^{f+2g}}{u}u^i\alpha_i \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4. \tag{2.24}$$

By using the formula $u^i\alpha_i = udu$ and that

$$u^i\overline{J}^i \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = udu \wedge \frac{\epsilon_{ijk}}{2}\alpha_i \wedge \alpha_j \wedge \overline{J}^k,$$

we can reexpress (2.24) as

$$\lambda\overline{J} \wedge \Phi = \lambda(e^{3f} - \frac{e^{f+2g}}{u^2})du \wedge \frac{\epsilon_{ijk}}{2}\alpha_i \wedge \alpha_j \wedge \overline{J}^k - 2\lambda e^{f+2g}du \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4. \tag{2.25}$$

By another side, the right side of the equation (2.23) is found directly from (3.41) and (2.13), the result is

$$d(e^h * \Phi) = (e^{2(f+g)+h})'du \wedge \frac{\epsilon_{ijk}}{2}\alpha_i \wedge \alpha_j \wedge \overline{J}^k + \Big((e^{4g+h})' - 4ue^{2(f+g)+h}\Big)du \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4. \tag{2.26}$$

By equating (2.26) with (2.25) and taking into account the closure condition (2.21) we obtain the following differential system

$$ue^{3f} = (e^{f+2g})', \qquad \lambda(e^{3f} - \frac{e^{f+2g}}{u^2}) = (e^{2(f+g)+h})',$$

$$4ue^{2(f+g)+h} - 2\lambda e^{f+2g} = (e^{4g+h})'. \tag{2.27}$$

From this system of equations we obtain the proposition stated above.

2.1 A particular solution: the Swann bundle

If we were able to solve the system (2.27) then we will obtain a family of $Spin(7)$ holonomy metrics based on arbitrary quaternion Kähler spaces. We do not know its general solution, but we have found a particular one. It is not difficult to check that indeed

$$e^f = u^{-1/3}, \qquad e^g = u^{2/3}, \qquad e^h = \lambda u^{-2/3}, \tag{2.28}$$

is a solution of (2.27). By introducing it into the expression (2.14) and by defining the variable $\tau = t/\lambda$ and rescaling by $g_8 \to \lambda^{-1} g_8$ we obtain the following metric

$$g_8 = u(d\tau + H)^2 + \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{u} + u g_q, \tag{2.29}$$

and the corresponding closed four form is

$$\Phi_4 = (d\tau + H) \wedge \left(\frac{\alpha_1 \wedge \alpha_2 \wedge \alpha_3}{u} + u \alpha_i \wedge \overline{J}_i \right) + \frac{\epsilon_{ijk}}{2} \alpha_i \wedge \alpha_j \wedge \overline{J}_k + u^2 e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

The closure of this form follows directly from formulas (2.13). An inspection of this metric shows that it is a cone over an Einstein-Sassaki metric, thus the holonomy is in $SU(4) \in G_2$. In order to see this we need to show the orthogonality condition $\widetilde{u}_i \theta_i = 0$, which is a consequence of the following calculation

$$\widetilde{u}_i \theta_i = \widetilde{u}_i d\widetilde{u}_i + \epsilon^{ijk} \widetilde{u}^i \omega_-^j \widetilde{u}^k = \widetilde{u}_i d\widetilde{u}_i = d(\widetilde{u}_i \widetilde{u}_i) = 0,$$

we have used $\widetilde{u}_i \widetilde{u}_i = 1$ in the last line. We also have that

$$u\theta^i + \frac{u_i du}{u} = \alpha_i,$$

and by inserting this expression in (2.29), applying after the orthogonality condition and by defining the new radius $r = 2u^{1/2}$ gives the following conical form of the metric

$$g_8 = dr^2 + r^2 g_7, \tag{2.30}$$

being g_7 given by

$$g_7 = (d\tau + H)^2 + g_6 = (d\tau + H)^2 + \theta_i \theta_i + g_q. \tag{2.31}$$

We have seen that the six dimensional metric g_6 is Kähler-Einstein and therefore g_7 is Einstein-Sassaki (see the lectures [33] and references therein). Any cone over an Einstein-Sassaki space is Calabi-Yau and therefore its holonomy is in $SU(4) \subset Spin(7)$.

But more information about these metrics can be found by finding explicitly the one form H, which is defined by $dH = \overline{J}$. In order to solve $dH = \overline{J}$ we need to simplify the expression (2.10). Let us remind that

$$\overline{J} = \widetilde{u}_i \overline{J}_i - \frac{\epsilon_{ijk}}{2} \widetilde{u}_i \theta_j \wedge \theta_k$$

and that $\theta_i = d(\widetilde{u}^i) + \epsilon^{ijk} \omega_-^j \widetilde{u}^k$. The orthogonality condition $\widetilde{u}_i \theta_i = 0$ is equivalent to

$$\theta_3 = -\frac{(\widetilde{u}_1 \theta_1 + \widetilde{u}_2 \theta_2)}{\widetilde{u}_3}.$$

From the last relation it follows that

$$\frac{\epsilon_{ijk}}{2}\widetilde{u}_i\theta_j\wedge\theta_k=\frac{\theta_1\wedge\theta_2}{\widetilde{u}_3}.$$

After certain calculation we obtain

$$\frac{\theta_1\wedge\theta_2}{\widetilde{u}_3}=\frac{d\widetilde{u}_1\wedge d\widetilde{u}_2}{\widetilde{u}_3}-d\widetilde{u}_i\wedge\omega_-^i+\frac{\epsilon^{ijk}}{2}\widetilde{u}_i\omega_-^j\wedge\omega_-^k.$$

Therefore

$$\frac{\epsilon_{ijk}}{2}\widetilde{u}_i\theta_j\wedge\theta_k=\frac{d\widetilde{u}_1\wedge d\widetilde{u}_2}{\widetilde{u}_3}-d\widetilde{u}_i\wedge\omega_-^i+\frac{\epsilon^{ijk}}{2}\widetilde{u}_i\omega_-^j\wedge\omega_-^k. \tag{2.32}$$

By another side we have the fundamental relation for quaternion Kahler manifolds, which is

$$\widetilde{J}_i=d\omega_-^i+\frac{\epsilon^{ijk}}{2}\omega_-^j\wedge\omega_-^k. \tag{2.33}$$

Inserting expressions (2.32) and (2.33) into (2.10) give us a remarkably simple expression for $\overline{J}$, namely

$$\overline{J}=d(\widetilde{u}_i\omega_-^i)-\frac{d\widetilde{u}_1\wedge d\widetilde{u}_2}{\widetilde{u}_3} \tag{2.34}$$

By parameterizing the coordinates $\widetilde{u}_i$ in the spherical form

$$\widetilde{u}_1=\cos\theta,\qquad \widetilde{u}_2=\sin\theta\cos\varphi,\qquad \widetilde{u}_3=\sin\theta\sin\varphi,$$

we find out that

$$\frac{d\widetilde{u}_1\wedge d\widetilde{u}_2}{\widetilde{u}_3}=-d\varphi\wedge d\cos\theta.$$

With the help of the last expression can reexpress (2.34) as

$$\overline{J}=d(\widetilde{u}_i\omega_-^i)-d\varphi\wedge d\cos\theta,$$

from where it follows directly that the form H such that $dH=\overline{J}$ is given by

$$H=u_i\omega_-^i+\cos\theta d\varphi, \tag{2.35}$$

up to a total differential term. By introducing the expression (2.35) into (2.31) we find directly the following expression for the Einstein-Sasaki metric

$$g_7=(d\tau+\cos\theta d\varphi+\cos\theta\omega_-^1+\sin\theta\cos\varphi\omega_-^2+\sin\theta\sin\varphi\omega_-^3)^2+(d\theta-\sin\varphi\omega_-^2+\cos\varphi\omega_-^3)^2$$
$$+(\sin\theta d\varphi+\sin\theta\omega_-^1-\cos\theta\cos\varphi\omega_-^2-\cos\theta\sin\varphi\omega_-^3)^2+g_q. \tag{2.36}$$

Let us introduce the coordinates u_i written in spherical form

$$u_1=|u|\sin\theta\cos\varphi\cos\phi,$$

$$u_2=|u|\sin\theta\cos\varphi\sin\phi,$$

$$u_3=|u|\sin\theta\sin\varphi,$$

$$u_4=|u|\cos\theta,$$

Then it is not difficult to check that the cone $g_8 = dr^2 + r^2 g_7$ being g_7 given at (2.36), can be expressed as

$$g_s = g|u|^2\overline{g} + f[(du_0 - u_i\omega_-^i)^2 + (du_i + u_0\omega_-^i + \epsilon_{ijk}u_k\omega_-^k)^2]. \tag{2.37}$$

The coordinates u_i can be extended to a single quaternion valued coordinate

$$u = u_0 + u_1 I + u_2 J + u_3 K, \qquad \overline{u} = u_0 - u_1 I - u_2 J - u_3 K.$$

Here I, J, K denote the unit quaternions, and it follows that $|du|^2 = (du_0)^2 + (du_1)^2 + (du_2)^2 + (du_3)^2$. The $Sp(1) \sim SU(2)$ triplet ω_-^i can be used to define a quaternion valued one form

$$\omega_- = \omega_-^1 I + \omega_-^2 J + \omega_-^3 K,$$

and the Kahler triplet $\overline{J}^a$ can be extended to a quaternion valued two form $\overline{J} = \overline{J}^1 I + \overline{J}^2 J + \overline{J}^3 K$. The metric (2.37) can be expressed in this notation as

$$g_8 = |u|^2 g_q + |du + u\omega_-|^2, \tag{2.38}$$

Under the transformation $u \to Gu$ with $G : M \to SU(2)$ the $SU(2)$ instanton ω_- is gauge transformed as $\omega_- \to G\omega_- G^{-1} + GdG^{-1}$. Therefore the form $du + \omega_- u$ is transformed as

$$du + u\omega_- \to d(Gu) + (G\omega_- G^{-1} + GdG^{-1})Gu = Gdu + (dG + G\omega_- - dG)u = G(du + u\omega_-),$$

and it is seen that $du + \omega_- u$ is a well defined quaternion-valued one form over the chiral bundle. Associated to the metric (2.38) we have the quaternion valued two form

$$\widetilde{\overline{J}} = u\overline{J}\overline{u} + (du + u\omega_-) \wedge \overline{(du + u\omega_-)}, \tag{2.39}$$

and it can be checked that the metric (2.37) is hermitic with respect to any of the components of (2.39). We have that

$$d\widetilde{\overline{J}} = du \wedge (\overline{J} + d\omega_- - \omega_- \wedge \omega_-)\overline{u} + u \wedge (\overline{J} + d\omega_- - \omega_- \wedge \omega_-)d\overline{u}$$

$$+u(d\overline{J} + \omega_- \wedge d\omega_- - d\omega_- \wedge \omega_-)\overline{u}.$$

The first two terms of the last expression are zero due to (2.8). Also by introducing (2.8) into the relation (2.6) we obtain that

$$d\overline{J} + \omega_- \wedge d\omega_- - d\omega_- \wedge \omega_- = 0$$

and therefore the third term is also zero. This means that the metric (2.38) is hyperkahler with respect to the triplet $\widetilde{\overline{J}}$ and the holonomy is reduced to $Sp(2) \subset SU(4) \subset Spin(7)$.

The hyperkahler metrics (2.38) are indeed well known. They are the Swann principal $CO(3)$ bundle of co-frames over a quaternion Kähler spaces [32]. Hyperkahler quotients of such metrics by tri-holomorphic isometries are related to quaternion Kahler quotients of the base spaces. The hyperkahler condition for the Swann metric implies that the seven dimensional cone of (2.31) is not only Einstein-Sassaki, but tri-Sasaki. The vector field ∂_ϕ is the Reeb vector of the tri-Sasaki metric.

The self duality of the spin connection

Although we have found that our example is hyperkahler, it is instructive to check that the spin connection ω_{ab} of the metric (2.29) is self-dual. We choose the basis

$$\overline{e}^{\alpha} = u^{1/2}e^{\alpha}, \qquad \overline{e}^{i} = \frac{\alpha^i}{u^{1/2}}, \qquad \overline{e}^{8} = u^{1/2}(d\tau + H)$$

being e^{α} a basis for g_q with $\alpha = 1,2,3,4$ and i $= 1,2,3$. With the help of the first Cartan equation

$$d\overline{e}^{m} + \hat{\omega}_{mn} \wedge \overline{e}^{n} = 0,$$

where m an index that can be latin or greek, we obtain the following components of the spin connection

$$\hat{\omega}_{ij} = \frac{u^{[i}\alpha^{j]}}{2u^2} - \epsilon_{ijk}\omega_{-}^{k} + \epsilon_{ijk}\frac{u_k}{u}(d\tau + H),$$

$$\hat{\omega}_{\alpha\beta} = \omega_{\alpha\beta} - \epsilon_{ijk}\frac{u_k}{u^2}(\overline{J}_j)_{\alpha\beta}\alpha^i - \frac{u_i}{u}(\overline{J}_i)_{\alpha\beta}(d\tau + H),$$

$$\hat{\omega}_{\alpha i} = \frac{u^i}{2}e^{\alpha} - \epsilon_{ijk}\frac{u_k}{u}(\overline{J}_j)_{\alpha\beta}e^{\beta},$$

$$\hat{\omega}_{8i} = \frac{u_i}{u}(d\tau + H) + \epsilon_{ijk}\frac{u_j}{u^2}\alpha^k,$$

$$\hat{\omega}_{8\alpha} = \frac{u_i}{u^2}(\overline{J}_j)_{\alpha\beta}e^{\beta}.$$

By using that $2\omega_{-}^{i} = \omega_{\alpha\beta}(\overline{J}_j)_{\alpha\beta}$ and the representation

$$J^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J^3 = J^1 J^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad (2.40)$$

for the matrix $(\overline{J}_j)_{\alpha\beta}$, it can be checked that

$$\hat{\omega}_{81} = -(\hat{\omega}_{23} + \hat{\omega}_{65} + \hat{\omega}_{47}), \qquad \hat{\omega}_{82} = -(\hat{\omega}_{31} + \hat{\omega}_{46} + \hat{\omega}_{57}),$$

$$\hat{\omega}_{83} = -(\hat{\omega}_{12} + \hat{\omega}_{54} + \hat{\omega}_{67}), \qquad \hat{\omega}_{84} = -(\hat{\omega}_{62} + \hat{\omega}_{35} + \hat{\omega}_{71}),$$

$$\hat{\omega}_{85} = -(\hat{\omega}_{16} + \hat{\omega}_{43} + \hat{\omega}_{72}), \qquad \hat{\omega}_{86} = -(\hat{\omega}_{15} + \hat{\omega}_{24} + \hat{\omega}_{73}),$$

$$\hat{\omega}_{87} = -(\hat{\omega}_{14} + \hat{\omega}_{36} + \hat{\omega}_{25}).$$

These conditions for $\hat{\omega}_{mn}$ can be written more concisely as

$$\hat{\omega}_{8i} = -c_{imn}\hat{\omega}_{mn},$$

and this is equivalent to say that, for the basis $\overline{e}^{m}$, the spin connection $\hat{\omega}_{mn}$ is self-dual.

3. Discussion

Along this brief work we have proposed an anzatz for $Spin(7)$ metrics as an R bundle over closed G_2 structures. These G_2 structures are R^3 bundles over 4-dimensional compact quaternion Kähler spaces. We also have used the fact that the twistor space of any compact quaternion Kähler space is Kähler Einstein and therefore there it possess a six dimensional sympletic form defined over it. We have imposed the conditions for the reduction of the holonomy to $Spin(7)$ and we have found a non linear system relating three unknown functions. We have found a particular solution and the result was the Swann bundle in eight dimensions, which is hyperkahler and therefore the holonomy is $Sp(2) \subset Spin(7)$. Let us recall that the direct sum

$$g_{11} = g_{1,2} + g_8,$$

of the Swann metric with the three dimensional flat Minkowski one $g_{1,2}$ is a solution of the supergravity equations of motion with all the fields "turned off" except the graviton, and preserving four supersymmetries after compactification. This solution can be rewritten in the IIA form

$$g_{11} = e^{-\frac{2}{3}\phi} g_{10} + e^{\frac{4}{3}\phi}(d\tau + H)^2,$$

being the dilaton ϕ defined by $\phi = \frac{3}{4}\log u$. The reduction along the isometry ∂_τ will give a background of the form

$$g_{IIA} = u^{1/2} g_{1,2} + u^{-1/2}\widetilde{g}_7.$$

being $\widetilde{g}_7$ given by

$$\widetilde{g}_7 = \frac{(du^1 + \omega_-^2 u^3)^2}{u^{1/2}} + \frac{(du^2 + \omega_-^3 u^1)^2}{u^{1/2}} + \frac{(du^3 + \omega_-^1 u^2)^2}{u^{1/2}} + u^{3/2} g_q.$$

The last metric together with the 3-form

$$\Phi = \frac{1}{u^{3/4}}\alpha_1 \wedge \alpha_2 \wedge \alpha_3 + u^{5/4}\alpha_i \wedge \overline{J}_i$$

and its dual

$$*\Phi = u\frac{\epsilon_{ijk}}{2}\alpha_i \wedge \alpha_j \wedge \overline{J}_k + u^3 e_1 \wedge e_2 \wedge e_3 \wedge e_4, \tag{3.41}$$

constitute a G_2 structure. Therefore and on general grounds we have that

$$d\Phi = \tau_o \wedge *\Phi + 3\tau_1 \wedge \Phi + *\tau_3,$$

$$d * \Phi = 4\tau_1 \wedge *\Phi + \tau_2 \wedge \Phi$$

being τ_i the four torsion classes. We have calculated them for our case and we have found that $\tau_3 = \tau_0 = 0$ and that the class τ_2 can be expressed entirely in terms of the sympletic form $\overline{J}$ for the Kähler-Einstein metric. The class τ_0 can be eliminated by a conformal transformation. The expression for the Swann metric (2.31) in terms of $\widetilde{g}_7$ is

$$g_8 = u(d\tau + H)^2 + \frac{1}{u^{1/2}}\widetilde{g}_7.$$

We can therefore paraphrase the results described in this work by stating that the Swann bundle defines a conformally closed G_2 structure with $\tau_3 = \tau_0 = 0$ by reduction along one isometry (which should not be confused with an hyperkahler reduction or quotient). It will be interesting to see if it is possible to find a one parameter deformation of the solution presented here and to see if the holonomy group obtained is bigger than $Sp(2)$. In our opinion, this question deserve some attention.

Acknowledgement: I am benefited with discussions with G. Giribet, who checked some calculations. Also my acknowledgments to D. Joyce who pointed out that the example presented here could be the Swann bundle and to S.Salamon for presenting me useful literature to compare.

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